Lesson 3: Applications of Exponential Growth or Decay

Applications of Exponential and Logarithmic Functions
Lesson #3: Applications of Exponential Growth or Decay

Review

- An exponential function is a function whose equation is of the form
  \[ y = ab^x \], where \( a \neq 0, b > 0, x \in \mathbb{R} \).
- For \( b > 1 \), the function represents a growth function.
- For \( 0 < b < 1 \), the function represents a decay function.

Writing an Equation Using \( y = ab^x \)

There are many applications of exponential functions in real life. In some cases, the function \( y = ab^x \) can be written in a “disguised” form. For example, \( A = P(1 + r)^t \) is an exponential function whose base is \( 1 + r \). In this lesson, we will meet further real life applications of exponential growth and decay.

We can use variations of the formula \( y = ab^x \) (such as \( A = P(1 + r)^t \)) to solve problems involving population growth, growth of bacteria, radioactive decay etc.

In 2012, the university population of a country was 160,000 and was increasing at an annual rate of 4.5%.

a) If the function representing the population is of the form \( y = ab^x \), state the values for \( a \) and \( b \).

\[ a = 160,000 \quad \text{b} = 1.045 \]

b) Write an equation to represent the university population, \( P \), of the country as a function of the number of years, \( n \), since 2012.

\[ P = 160,000 \times (1.045)^n \]

c) Determine the population in the year 2015.

\[ n = 3 \quad \Rightarrow \quad P = 160,000 \times (1.045)^3 = 192,586 \]

b) If the population continues to grow at this rate, determine the number of years, to the nearest year, for the population to double from its 2012 size.

\[ 320,000 = 160,000 \times (1.045)^n \]

\[ 2 = (1.045)^n \]

\[ \log_{1.045} 2 = n \]

\[ \frac{\log 2}{\log 1.045} = n \]

16 years to double.
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Class Ex. #2

The number of fish in a lake is decreasing by 5% each year as a result of overfishing.

a) If the number of fish present is an exponential function of time, state the base of the exponential function.

\[ N(t) = N_0(0.95)^t \]

b) Write an equation to represent the number of fish present after \( t \) years.

Use \( N_0 \) to represent the initial population and \( N(t) \) to represent the final population.

\[ N(t) = 2500(0.95)^t \]

\[ N(2012 - 2017 = 5 \text{ years}) = t \]

\[ N(5) = 2500(0.95)^5 = 1934 \text{ fish} \]

c) If there were 2500 fish present in June 2012, how many would you expect to be present in June 2017?

\[ N(5) = 2500(0.95)^5 = 1934 \text{ fish} \]

\[ t = \frac{\log 0.95}{\log 0.5} \]

\[ t = 13.51... \]

13.5 years

d) How many years, to the nearest tenth, would it take for the fish population to reduce to half of the number in June 2012?

\[ N(t) = 2500(0.95)^t \]

\[ 0.5 = 0.95^t \]

\[ \log 0.5 = t \]

\[ t = 13.51... \]

13.5 years

Class Ex. #3

The intensity, \( I_0 \), of a light source is reduced to \( I \) after passing through \( d \) metres of a fog, according to the formula \( I = I_0 e^{-0.12d} \). At what distance, to the nearest hundredth of a metre, will the intensity be reduced to one quarter of its original value?

\[ I = \frac{1}{4} I_0 \]

\[ I_0 e^{-0.12d} = \frac{1}{4} I_0 \]

\[ e^{-0.12d} = \frac{1}{4} \]

\[ -0.12d = \ln \left( \frac{1}{4} \right) \]

\[ -0.12d = \ln \left( \frac{1}{4} \right) \]

\[ d = \frac{\ln \left( \frac{1}{4} \right)}{-0.12} \]

\[ d = 11.55... \]

11.55 m

Complete Assignment Questions #1 - #7

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Developing an Exponential Formula for Half-Life, Doubling Time, etc.

**Half-Life**

Radioactive substances decay by changing into non-radioactive substances. The amount of radioactive substance present at any given time is an exponential function of time.

After an interval of time, $H$, called the half-life of the radioactive substance, only half the original amount will still be radioactive.

Let $A$ represent the amount of a radioactive substance present at time, $t$, and let $A_0$ represent the amount present initially.

For some base, $b$, we can write the exponential function of time as $A = A_0b^t$.

Complete the following to determine the base in terms of the half-life:

\[
\begin{align*}
A &= A_0b^t \\
\frac{1}{2}A_0 &= A_0b^H \\
\frac{1}{2} &= b^H \\
\frac{1}{2} &= b^H \\
\ln\left(\frac{1}{2}\right) &= H \ln b \\
H &= \frac{\ln\left(\frac{1}{2}\right)}{\ln b} \\
A &= A_0\left(\frac{1}{2}\right)^\frac{t}{H}.
\end{align*}
\]

The exponential function is now of the form $A = A_0\left(\frac{1}{2}\right)^\frac{t}{H}$, which can be written as $A = A_0\left(\frac{1}{2}\right)^\frac{t}{H}$.

**Doubling Time**

The time it takes for an exponential function to double in value is called the doubling time.

The base in the formula for doubling time is $2^{\frac{1}{D}}$, where $D$ is the doubling time, and the exponential function is of the form $A = A_0\left(2^{\frac{1}{D}}\right)^t$, which can be written as $A = A_0\left(2^{\frac{1}{D}}\right)^t$.

Use a similar procedure to the one above to develop this formula.

\[
\begin{align*}
\frac{A}{A_0} &= \left(2^{\frac{1}{D}}\right)^t \\
\frac{A}{A_0} &= \left(2^{\frac{1}{D}}\right)^t \\
\frac{A}{A_0} &= \left(2^{\frac{1}{D}}\right)^t \\
\frac{A}{A_0} &= \left(2^{\frac{1}{D}}\right)^t \\
A &= A_0\left(2^{\frac{1}{D}}\right)^t.
\end{align*}
\]

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The formulas on the previous page are examples of the following general formula which can be used for solving problems involving doubling time, tripling time, half-life, etc. 

\[ A = A_0e^{Ct} \]

where, 
- \( A_0 \) = initial amount 
- \( A \) = amount at time \( t \) 
- \( C \) = constant, for example, 2 for doubling, 3 for tripling, \( \frac{1}{2} \) for half life 
- \( t \) = time 
- \( p \) = period of time for doubling, tripling, halving, etc.

Note that the base of the exponential function is \( e \).

\[ p = \frac{\ln(2)}{C} \]

\[ p = \frac{\log_2(2)}{C} \]

\[ p = \frac{\ln(3)}{C} \]

\[ p = \frac{\log_3(3)}{C} \]

\[ p = \frac{\ln(\frac{1}{2})}{C} \]

\[ p = \frac{\log_2(\frac{1}{2})}{C} \]

\[ p = \frac{\ln(0.03)}{C} \]

\[ p = \frac{\log_3(0.03)}{C} \]

\[ p = \frac{\ln(0.02)}{C} \]

\[ p = \frac{\log_3(0.02)}{C} \]

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A radioactive isotope has a half-life of 7 years.

(a) Use the formula \( A = A_0 C^t \), with \( C = \frac{1}{2} \), to determine how much of the isotope must initially be present to decay to 60 grams in 14 years.

\[
\frac{60}{A_0} = \left(\frac{1}{2}\right)^{\frac{14}{7}}
\]

(b) Write an equivalent formula with \( C = 2 \) which would solve the problem in (a).

(c) In this particular example, explain how the solution to the problem in (a) could be found without using any formula.

Since we know that the half-life is 7 years, and we know that there are 60 grams left after 14 years, the final (60 grams) is \( \frac{1}{4} \) of the original amount. => Original amount = 60 \( \times \) 4 = 240 grams.

### Assignment

1. A truck bought for $35,000 depreciates at a rate of 12% per year.

   (a) If the value of the truck is an exponential function of time, state the base of the exponential function.

   (b) Write an equation to represent the value, \( V \), of the truck after \( t \) years.

   (c) Determine, to the nearest hundred dollars, the value of the truck after 4 years.

   (d) How many years, to the nearest tenth, would it take for the value of the truck to reduce to one quarter of its purchase price?

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2. In 2008 the world population was approximately 6.7 billion and was increasing at an annual rate of 1.3%.
   a) If the function representing the population, in billions, is of the form $y = ab^n$, state values for $a$ and $b$.

   b) Write an equation to represent the world population, $W$ billions, as a function of the number of years, $n$, since 2008.

   c) Assuming the same growth rate, determine, to the nearest tenth of a billion, the expected world population in the year 2025.

   d) If the population continues to grow at this rate, determine the number of years, to the nearest year, for the population to double from its 2008 size.

   e) Estimate the world population in 1950. State any assumptions you have made. How does your answer compare with the actual world population in 1950? Give a reason for any discrepancy.

3. The value of a type of robotic technology depreciates 25% per year.
   a) Write an exponential function to represent the value of this robotic technology after $n$ years.

   b) How many years, to the nearest year, would it take for the value of the robotic technology, which initially cost $575,000, to depreciate to $25,000?
4. A town in southern British Columbia is growing at a rate of 3.5% per annum. If the town continues to grow at this rate, it is projected that the population will reach 20 000 in 5 years.
   a) Determine, to the nearest ten people, the current population of the town.
   
b) Assuming the same growth rate, determine how many years from now the population will reach 30 000. Answer to the nearest year.

5. A quantity of water contains 500 g of pollutants. Each time the water passes through a filter, 18% of the pollutants are removed. How many filters are needed to reduce the mass of pollutants to less than 150 g?

6. An x-ray beam of intensity, $I_0$, passing through absorbing material $x$ millimetres thick, emerges with an intensity $I$, given by $I = I_0e^{-kx}$. When the material is 9 millimetres thick, 50% of the intensity is lost.
   a) Calculate the value of the constant $k$ to the three decimal places.
   
b) What percentage intensity, to one decimal place, remains if the material is 20 millimetres thick?

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7. A hot piece of metal loses heat according to the formula \( T = T_0e^{-0.2t} \), where \( T \) is the temperature difference between the metal and the surrounding air after \( t \) minutes and \( T_0 \) is the initial temperature difference.

a) If the initial temperature of the metal was 330°C and of the air 30°C, find the temperature of the metal, to the nearest degree, after 5 minutes.

b) A different piece of hot metal cools to a temperature of 200°C after 8 minutes. What was the original temperature of the metal, to the nearest degree, if the air temperature was 27°C?

8. How much of a radioactive substance must be present to decay to 30 grams in 12 years if the half-life of the substance is 5.2 years? Round the answer to the nearest gram.

9. A radioactive isotope has a half-life of approximately 45 minutes. How long would it take for 480 mg of the isotope to decay to 15 mg?
10. A lab technician placed a bacterial cell into a vial at 5 a.m. The cells divide in such a way that the number of cells doubles every 4 minutes. The vial is full one hour later.
   a) How long does it take for the cells to divide to produce 4096 cells?

b) At what time is the vial half full?

c) At what time is the vial \( \frac{1}{16} \) full?

11. The population of germs in a dirty bathtub doubles every 20 minutes. How long, to the nearest minute, would it take for the population to triple?

12. A radioactive isotope has a half-life of approximately 25 weeks. How much of a sample of 50 grams of the isotope would remain after 650 days? (Round the answer to the nearest hundredth of a gram.)

13. What is the half-life, to the nearest month, of a radioactive isotope if it takes 7 years for 500 grams to decay to 35 grams?

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14. The tripling period, to the nearest tenth of an hour, of a bacterial culture which grows from 500 cells to 64,000 cells in 50 hours is ______.

15. Radioactive material decays to 40% of its original mass in 5 years. The half-life of the radioactive material, to the nearest hundredth of a year, is ______.

**Answer Key**

1. a) 0.88  b) $V = 35,000(0.88)^t$  c) $V = 0$  d) 10.8

2. a) $g = 6.7$, $h = 1.013$  b) $W = 6.7(1.013)^t$  c) 8.3 billion  d) 54 years
   e) 3.2 billion assuming a growth rate of 1.3% since 1950. The actual population was 2.55 billion, so the average growth rate since 1950 must have been greater than 1.3%.

3. a) $t = \ln(0.75) / \ln(2)$  b) 11  c) 16 40  d) 17 years  e) 7 filters

4. a) 0.077  b) 21.4%  c) 140°C  d) 86°F  e) 149 grams

5. 225 min  10. a) 48 min  b) 5:56 a.m.  c) 5:44 a.m.

6. 32 min  12. 4.32 g  15. 21 months

7. 1 1 3 11. 9 7 8

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